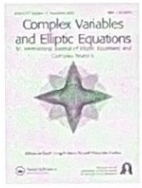
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1.	Riemann problem for multiply connected domain in Besov spaces	баспа	Boundary Value Problems 2024 (1), 70 (2024)	9	N Bliev
2.	Boundary conjugation problem for piecewise analytic functions in Besov spaces	баспа	Complex Variables and Elliptic Equations 69 (5), 849-856 (2024)	7	N Bliev
3.	Noetherian solvability and explicit solution of a singular integral equation with weighted Carleman shift in Besov spaces	баспа	Complex Variables and Elliptic Equations 68 (8), 1264-1285 (2023)	22	N Bliev, K Tulenov

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Complex Variables and Elliptic Equations
An International Journal

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/gcov20

Noetherian solvability and explicit solution of a singular integral equation with weighted Carleman shift in Besov spaces

N. K. Bliev, K. S. Tulenov & N. M. Yerkinbayev

To cite this article: N. K. Bliev, K. S. Tulenov & N. M. Yerkinbayev (2022): Noetherian solvability and explicit solution of a singular integral equation with weighted Carleman shift in Besov spaces, Complex Variables and Elliptic Equations, DOI: 10.1080/17476933.2022.2045975

To link to this article: https://doi.org/10.1080/17476933.2022.2045975

Published online: 12 Mar 2022.

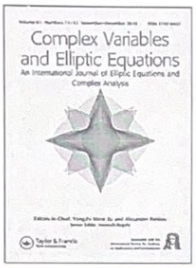
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To cite this article: N. K. Bliev & N. M. Yerkinbayev (2023): Boundary conjugation problem for piecewise analytic functions in Besov spaces, Complex Variables and Elliptic Equations, DOI: 10.1080/17476933.2023.2164887

To link to this article: https://doi.org/10.1080/17476933.2023.2164887

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Boundary conjugation problem for piecewise analytic functions in Besov spaces

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ABSTRACT
The main aim of this work is to show the solvability of the homogeneous and inhomogeneous boundary conjugation problems for piecewise analytic functions in the Besov space, which is embedded into the class of continuous functions.

ARTICLE HISTORY
Received 24 May 2022
Accepted 27 December 2022

COMMUNICATED BY
H. Begehr

KEYWORDS
Boundary value conjugation problems; piecewise-analytic functions; Cauchy type integrals; Besov spaces

AMS SUBJECT CLASSIFICATIONS
46E30; 47B10; 46L51; 46L52; 44A15; Secondary 47L20; 47C15

1. Introduction

Let Γ be a smooth simple closed contour, dividing the complex plane into an interior domain D^+, and an exterior domain D^-, including the point $z = \infty$.

For definiteness, the origin belongs to D^+, the positive direction of traversing the contour Γ leaves the region D^+ to the left.

The boundary conjugation problem (BCP) consists of finding the function $\Phi(z)$ which is analytic except on Γ, has finite order at ∞, and has limiting values $\Phi^{\pm}(t)$, $t \in \Gamma$, on both sides of Γ, that satisfy the conditions:

$$\Phi^+(t) = G(t)\Phi^-(t) \quad \text{(homogeneous problem)}, \tag{1}$$

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t) \quad \text{(inhomogeneous problem)} \tag{2}$$

where g and G are given functions.

In 1857, BCP for analytic functions was first obtained by Riemann [1]. In 1904, Hilbert reduced this problem to an integral equation, thereby giving the first proof of the existence of solution. At present, in some literature, the problem is referred as the Riemann and Riemann–Hilbert problems.

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At the same time, Plemelj showed that the solution of BCP can be obtained by using the Cauchy-type integral [2]. He considered only a special case when the index is equal to zero. This idea turned out to be so fruitful that even now the Cauchy-type integral is the main tool for investigating and solving boundary conjugation problems. The next stage in the development of this direction is associated with the names of Yu. Sokhotski, H. Poincaré, G. Bertrand, F. Noether, T. Carleman.

For the first time, a general solution of the problems (1) and (2) was given by Gakhov [3]. In the scalar case for general piecewise smooth curves $\Gamma = L$, these problems have been investigated by Muskhelishvili [4]. Further development of the theory of boundary conjugation problem is based on the modern theory of functions in the space of Hölder continuous functions and in Lebesgue space [5–11].

In this work, we study the problems (1) and (2) in Besov space. We emphasize that applications of Besov spaces possessing limit embedding theorems allow to weaken the condition from the density of Hölder continuous integrals by replacing it just continuity in terms of Besov spaces. Furthermore, the Besov space, we are considering, is embedded into the space of continuous functions on a closed Lyapunov contour but not into the class of functions satisfying Hölder condition [12]. This is the main reason which motivates us to study these problems in Besov spaces.

Typical examples of functions in Besov spaces in the circle $\Gamma = \mathbb{T} : |z| < d < 1$: for any fixed $\varepsilon > 0$, the functions

$$\frac{\varepsilon\, e^{i\varphi}}{2r \log^{1+\varepsilon} \frac{1}{r}} \in B_{p,1}^{\frac{2}{p}-1}(\mathbb{T}) \quad \text{and} \quad \frac{1}{\log^{\varepsilon} \frac{1}{r}} \in B_{p,1}^{\frac{2}{p}}(\mathbb{T}), \quad 1 < p < 2,\ z = r e^{i\varphi}.$$

Boundary value problems with operators with complex conjugate are one of the important parts of the theory of analytic functions of complex variables. Such problems have significant applications and they are playing a crucial role, for example, in the theory of boundary value problems for differential operators [13].

2. Preliminaries

2.1. *Besov space*

Let $t(s) = x(s) + iy(s)$, $0 \le s \le l$, be an equation of a contour $\Gamma \in C_\nu^m$, $m \ge 0$ an integer, $0 < \nu \le 1$, where $x(s)$, $y(s)$ are real-valued functions with period l extended to the entire real axis. Let $1 \le p \le \infty$, $1 \le \theta \le \infty$, $0 < r < m - 1 + \nu$ and k be an integer such that $k > r$. The Besov space $B_{p,\theta}^r(\Gamma)$ is the class of all functions $\varphi(t) = \varphi(t(s)) = \psi(s) \in L_p(\Gamma)$ with the finite seminorm:

$$\|\varphi\|_{b_{p,\theta}^r[0,l]} = \left(\int_0^l |h|^{-1-\theta r} \|\Delta_h^k \psi\|_{L_p[0,l]}^\theta \, dh \right)^{\frac{1}{\theta}} < \infty, \tag{3}$$

where $\Delta_h^k \psi(s) = \Delta_h(\Delta_h^{k-1}\psi)(s) = \sum_{l=0}^k (-1)^{k-l} C_k^l \psi(s + lh)$ is the difference of order k with step h. The space $B_{p,\theta}^r(\Gamma)$ becomes a Banach space when equipped with the norm

$$\|\varphi\|_{B_{p,\theta}^r(\Gamma)} = \|\psi\|_{L_p[0,l]} + \|\varphi\|_{b_{p,\theta}^r[0,l]}. \tag{4}$$

For more details on such spaces, we refer the reader to [12, 14, 15].

3. A general solution of the homogeneous conjugation problem

Throughout the paper, we denote by $\varkappa$ the index of $G(t)$, $t \in \Gamma$, and by $P_\varkappa(z)$, a polynomial of degree $\varkappa$ with arbitrary coefficients.

Let Γ be a closed Lyapunov contour from the class C^1_ν, $\frac{2}{p} - 1 < \nu \leq 1$, $1 < p < 2$.

Theorem 3.1: *Assume* $G(t) \in B^{\frac{1}{p}}_{p,1}(\Gamma)$, *and* $G(t) \neq 0$, $t \in \Gamma$, *and* $g(t) \in B^{\frac{1}{p}}_{p,1}(\Gamma)$. *Then*

(i) *if* $\varkappa \geq 0$, (1) *has* $\varkappa + 1$ *linearly independent solutions in* $B^{\frac{1}{p}}_{p,1}(\Gamma)$.
The general solution is given by

$$\Phi^+(z) = P_\varkappa(z)\, e^{\Pi^+(z)}, \quad \Phi^-(z) = z^{-\varkappa} P_\varkappa(z)\, e^{\Pi^-(z)}, \tag{5}$$

where

$$\Pi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\log\left[\tau^{-\varkappa} G(\tau)\right]}{\tau - z}\, d\tau. \tag{6}$$

(ii) *if* $\varkappa < 0$, (1) *is unsolvable.*

Proof: Let $\varphi(t) \in B^{\frac{1}{p}}_{p,1}(\Gamma)$, $1 < p < 2$. Then the Cauchy-type integral

$$\Phi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - z}\, d\tau, \quad z \in D^\pm \tag{7}$$

belongs to the space $B^{1+\alpha}_{p,1}(D^\pm)$, $\alpha = \frac{2}{p} - 1$, [12, Lemma 1.2].

Hence, $\Phi(z)$ has a trace on Γ from $B^{\frac{1}{p}}_{p,1}(\Gamma)$ [14], which matches with

$$\Phi^+(t) = \lim_{z \to t \in \Gamma} \Phi(z), \quad z \in D^+.$$

Therefore

$$\|\Phi^+\|_{B^{\frac{1}{p}}_{p,1}(\Gamma)} \leq M_1 \|\Phi\|_{B^{1+\alpha}_{p,1}(D^\pm)} \leq M_2 \|\varphi\|_{B^{\frac{1}{p}}_{p,1}(\Gamma)},$$

where M_1, M_2 are positive constants.

In our case, the formula

$$\Phi^+(t) - \Phi^-(t) = \varphi(t), \quad \varphi(t) \in B^{\frac{1}{p}}_{p,1}(\Gamma) \hookrightarrow C(\Gamma), \tag{8}$$

holds [4, p. 67] . With the additional condition $\Phi^-(\infty) = 0$, it follows that

$$\|\Phi^-\|_{B^{\frac{1}{p}}_{p,1}(\Gamma)} \leq \|\Phi^+\|_{B^{\frac{1}{p}}_{p,1}(\Gamma)} + \|\varphi\|_{B^{\frac{1}{p}}_{p,1}(\Gamma)} \leq M_3 \|\varphi\|_{B^{\frac{1}{p}}_{p,1}(\Gamma)},$$

where M_3 is a positive constant.

The Sokhotski–Plemelj formulas

$$\Phi^+(t) = \frac{1}{2}\varphi(t) + \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}\,d\tau, \tag{9}$$

$$\Phi^-(t) = -\frac{1}{2}\varphi(t) + \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}\,d\tau, \tag{10}$$

are valid, which follow from the results of [12, 16].

Let $\varkappa = 0$. Then $\log G(t)$ is a single-valued function.

Taking the logarithm of both sides of (1), we obtain

$$\log \Phi^+(t) - \log \Phi^-(t) = \log G(t). \tag{11}$$

We can choose any branch for $\log G(t)$. It is easy to check that the final result does not depend on the choice of the branch.

We have arrived at the problem of finding an analytic function $\log \Phi(z)$ by the jump given on Γ. Its solution, under an additional condition $\log \Phi^-(\infty) = 0$, is given by the formula

$$\log \Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\log G(\tau)}{\tau - z}\,d\tau. \tag{12}$$

For brevity, we denote $\Pi_0(z) = \log \Phi(z)$. The solutions of (1), satisfying the condition $\Phi^-(\infty) = 1$, are given by

$$\Phi^+(z) = e^{\Pi_0^+(z)}, \quad \Phi^-(z) = e^{\Pi_0^-(z)},$$

as follows from the Sokhotski–Plemelj formulas.

If we remove the additional condition $\Phi^-(\infty) = 1$, then the solution of (1) in this case is defined by

$$\Phi^+(z) = A\,e^{\Pi_0^+(z)}, \quad \Phi^-(z) = A\,e^{\Pi_0^-(z)},$$

where A is an arbitrary constant. Since $\Pi_0^-(\infty) = 0$, A represents the value of $\Phi^-(z)$ at infinity.

Thus, we conclude that if $\operatorname{Ind} G(t) = 0$, and $G(t) \neq 0$ with $G(t) \in B_{p,1}^{\frac{1}{p}}(\Gamma)$, then

$$G(t) = \frac{e^{\Pi_0^+(t)}}{e^{\Pi_0^-(t)}}, \quad \Pi_0(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\log G(\tau)}{\tau - z}\,d\tau.$$

Therefore, in the case $\varkappa = 0$ and for arbitrary $\Phi^-(\infty) \neq 0$, the solution contains one arbitrary constant, therefore, there is only one linearly independent solution. If $\Phi^-(\infty) = 0$, then $A = 0$ and the problem has only the trivial solution.

Let $\varkappa > 0$, assume that the origin belongs to D^+, then we have

$$\operatorname{Ind} t^{\varkappa} = \frac{1}{2\pi}\int_\Gamma \mathrm{d}\arg t^{\varkappa} = \frac{\varkappa}{2\pi}\int_\Gamma \mathrm{d}\arg t = \varkappa.$$

It follows that

$$\begin{aligned}\operatorname{Ind}[t^{-\varkappa}G(t)] &= \frac{1}{2\pi}\int_\Gamma \mathrm{d}\arg\left[t^{-\varkappa}G(t)\right] \\ &= \frac{1}{2\pi}\int_\Gamma \mathrm{d}\arg t^{-\varkappa} + \frac{1}{2\pi}\int_\Gamma \mathrm{d}\arg G(t) \\ &= -\varkappa + \varkappa = 0.\end{aligned}$$

Then we obtain

$$t^{-\varkappa}G(t) = \frac{\mathrm{e}^{\Pi^+(t)}}{\mathrm{e}^{\Pi^-(t)}}, \quad \Pi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\log\left[\tau^{-\varkappa}G(\tau)\right]}{\tau - z}\,\mathrm{d}\tau.$$

It is easy to obtain an expression for the boundary values of the canonical function:

$$\chi(z) = \begin{cases}\chi^+(z) = \mathrm{e}^{\Pi(z)}, & \text{if } z \in D^+, \\ \chi^-(z) = z^{-\varkappa}\mathrm{e}^{\Pi(z)}, & \text{if } z \in D^-,\end{cases} \tag{13}$$

Then the function $G(t)$ from (1) can be represented as

$$G(t) = \frac{\mathrm{e}^{\Pi^+(t)}}{t^{-\varkappa}\mathrm{e}^{\Pi^-(t)}} = \frac{\chi^+(t)}{\chi^-(t)}. \tag{14}$$

The canonical function $\chi(z)$ is a function that satisfies the boundary condition (1) and is analytic everywhere in the plane, with the exception of the infinitely distant point, where its order is equal to the index of the problem. This function can be constructed when the index of problem is zero.

Representing $G(t)$ by formula (14), we rewrite (1) in the form

$$\frac{\Phi^+(t)}{\chi^+(t)} = \frac{\Phi^-(t)}{\chi^-(t)}.$$

The last equality shows that $\frac{\Phi^+(t)}{\chi^+(t)}$ is analytic in D^+, and $\frac{\Phi^-(t)}{\chi^-(t)}$ is analytic in D^-, with the exception of the point at infinity, where it can have a pole of order at most $\varkappa$. Note both $\frac{\Phi^+(t)}{\chi^+(t)}$ and $\frac{\Phi^-(t)}{\chi^-(t)}$ are analytic continuations of each other through the contour Γ [3, Chapter 12]. Consequently, they are branches of a single analytic function, which in the entire plane can have a single singularity, it means the pole of order at most $\varkappa$ at infinity. By the generalized Liouville theorem [3, Chapter 12], this function is a polynomial of degree at most $\varkappa$ with arbitrary complex coefficients.

For $\varkappa > 0$, we obtain the solution of (1) in the form

$$\Phi(z) = P_\varkappa(z)\chi(z)$$

or

$$\Phi^+(z) = P_\varkappa(z)\,\mathrm{e}^{\Pi^+(z)}, \quad \Phi^-(z) = z^{-\varkappa}P_\varkappa(z)\,\mathrm{e}^{\Pi^-(z)}.$$

Let $\varkappa < 0$ then the canonical function has a pole of order $|\varkappa|$ at infinity and is no longer a solution, but it will be used as an auxiliary function in solving an inhomogeneous problem.

This completes the proof. ∎

4. A general solution of the inhomogeneous conjugation problem

Theorem 4.1: *Let functions $G(t)$ and $g(t)$ belong to $B^{\frac{1}{p}}_{p,1}(\Gamma)$, and $G(t) \neq 0$, $t \in \Gamma$. Then*

(i) if $\varkappa \geq 0$, (2) is solvable, and general solution is given by

$$\Phi(z) = \frac{\chi(z)}{2\pi i}\int_\Gamma \frac{g(\tau)}{\chi^+(\tau)}\frac{\mathrm{d}\tau}{\tau - z} + P_\varkappa(z)\chi(z), \tag{15}$$

where $\chi(z)$ is the canonical function and is determined by (13).

(ii) if $\varkappa = -1$, then the problem (2) is also solvable and has a unique solution.

(iii) if $\varkappa < -1$, the problem (2) is unsolvable. In order (2) to be solvable, it is necessary and sufficient that the free term of the problem satisfies the $-\varkappa - 1$ conditions

$$\int_\Gamma \frac{g(\tau)}{\chi^+(\tau)}\tau^{k-1}\,\mathrm{d}\tau = 0, \quad (k = 1, 2, \ldots, -\varkappa - 1). \tag{16}$$

Then the unique solution of (2) is given by formula (15), where $P_\varkappa(z) \equiv 0$.

Proof: Replacing the coefficient $G(t)$ of the boundary condition (2) by (14), we rewrite the boundary condition (2) in the form

$$\frac{\Phi^+(t)}{\chi^+(t)} = \frac{\Phi^-(t)}{\chi^-(t)} + \frac{g(t)}{\chi^+(t)},$$

where $\frac{g(t)}{\chi^+(t)} \in B^{\frac{1}{p}}_{p,1}(\Gamma)$. Then using the formula (8), we replace it by the difference of the boundary values of analytic functions:

$$\frac{g(t)}{\chi^+(t)} = \Psi^+(t) - \Psi^-(t),$$

where

$$\Psi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{g(\tau)}{\chi^+(\tau)}\frac{\mathrm{d}\tau}{\tau - z}. \tag{17}$$

Then the boundary condition (2) can be written in the form

$$\frac{\Phi^+(t)}{\chi^+(t)} - \Psi^+(t) = \frac{\Phi^-(t)}{\chi^-(t)} - \Psi^-(t).$$

The solutions of (2) is given depending on the cases:

Case $\varkappa \geq 0$. Then

$$\frac{\Phi^+(z)}{\chi^+(z)} - \Psi^+(z) = \frac{\Phi^-(z)}{\chi^-(z)} - \Psi^-(z) = P_\varkappa(z).$$

It follows that

$$\Phi(z) = \chi(z)\Psi(z) + \chi(z)P_\varkappa(z). \tag{18}$$

Case $\varkappa < 0$. For this case we have $P_\varkappa(z) \equiv 0$, and obtain

$$\Phi(z) = \chi(z)\Psi(z). \tag{19}$$

The first factor $\chi^-(z)$ of the function $\Phi^-(z)$ has a pole of the order $-\varkappa$ at infinity, and the second, as a Cauchy-type integral (17), has a zero of first order at infinity. Therefore, $\Phi^-(z)$ has a pole of order not greater than $-\varkappa - 1$ at infinity. Thus, if $\varkappa < -1$, problem (2) is unsolvable. It will be solvable only when the free term satisfies some additional conditions. To obtain them, we decompose the Cauchy-type integral (17) in a neighbourhood of infinity [3, see Chapter1] in a series

$$\Psi^-(z) = \sum_{k=1}^{\infty} c_k z^{-k},$$

where

$$c_k = -\frac{1}{2\pi i}\int_\Gamma \frac{g(\tau)\tau^{k-1}}{\chi^+(\tau)}\,\mathrm{d}\tau.$$

For analyticy of $\Phi^-(z)$ at infinity, it is necessary that the first $-\varkappa - 1$ coefficients of $\Psi^-(z)$ vanish. Hence, we obtain that for the solvability of problem (2) in the case of a negative index ($\varkappa < -1$) it is necessary and sufficient that $-\varkappa - 1$ conditions are satisfied:

$$\int_\Gamma \frac{g(\tau)\tau^{k-1}}{\chi^+(\tau)}\,\mathrm{d}\tau = 0, \quad k = 1, 2, \ldots, -\varkappa.$$

This completes the proof. ■

Acknowledgments

We thank the anonymous referee for reading the paper carefully and providing thoughtful comments, which improved the exposition of the paper.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This work was partially supported by Grant No. AP19677346 of the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan.

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